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Fekete-Szegö problem for certain subclasses of analytic functions with respect to symmetric points

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Let $S^n_{\lambda,\mu}(\alpha,t;\delta,A,B)$ be the class of normalized analytic functions defined in the open unit disk U satisfy the following subordination

$$(1-\alpha)\frac{(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}f(z)-D^n_{\lambda,\mu}f(tz)}+\alpha\frac{\left[(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'\right]'}{\left[D^n_{\lambda,\mu}f(z)-D^n_{\lambda,\mu}f(tz)\right]'}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta},$$

where $|t|\leq 1$, $t\neq 1$, $0\leq\alpha\leq 1$, $0<\delta\leq 1$, $-1\leq B<A\leq 1$ and $D^n_{\lambda,\mu}$ is the Raducanu-Orhan multiplier differential operator. The object of the present investigation is to obtain sharp upper bounds for the Fekete-Szegö functional $\left|a_3-\xi a_2^2\right|$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1.1}$$

that are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Let Ω denote the class of bounded analytic functions $w(z)$ in U satisfying the conditions $w(0)=0$ and $|w(z)|<1$ for $z\in U$. It is known [13] that

$$|c_1|\leq 1,\qquad |c_2|\leq 1-|c_1|^2. \tag{1.2}$$

Let us recall the principle of subordination between analytic functions. Let the functions f and g be analytic in U. Then we say that f is subordinate to g, if there exists a Schwarz function $w(z)$, analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in U),$$

such that

$$f(z) = g(w(z)) \quad (z \in U).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \ \ (z \in U).$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For $f(z)$ belonging to $\mathcal{A}$, *the multiplier differential operator* $D^n_{\lambda,\mu}f$ was defined by Raducanu and Orhan (see [19]) as follows:

$$D^0_{\lambda,\mu}f(z) = f(z),$$

$$D^1_{\lambda,\mu}f(z) = D_{\lambda,\mu}f(z) = \lambda\mu z^2(f(z))'' + (\lambda-\mu)z(f(z))' + (1-\lambda+\mu)f(z),$$

$$D^2_{\lambda,\mu}f(z) = D_{\lambda,\mu}\left(D^1_{\lambda,\mu}f(z)\right),$$

$$\vdots$$

$$D^n_{\lambda,\mu}f(z) = D_{\lambda,\mu}\left(D^{n-1}_{\lambda,\mu}f(z)\right),$$

where $0 \leq \mu \leq \lambda \leq 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Very recently, the operator $D^n_{\lambda,\mu}f$ was extended to $\lambda \geq \mu \geq 0$ by Deniz and Orhan in [4]. We use this operator for $\lambda \geq \mu \geq 0$.

If f is given by (1.1), then from the definition of the operator $D^n_{\lambda,\mu}f(z)$ it is easy to see that

$$D^n_{\lambda,\mu}f(z) = z + \sum_{k=2}^{\infty} \Phi^n_k a_k z^k,$$

where $\Phi^n_k = [1 + (\lambda\mu k + \lambda - \mu)(k-1)]^n, (\Phi^n_k = [\Phi_k]^n)$; $\lambda \geq \mu \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It should be remarked that $D^n_{\lambda,\mu}$ generalizes many other linear operators considered earlier by different authors. In particular, for $f \in \mathcal{A}$ we have the following:

- $D^n_{1,0}f(z) \equiv D^n f(z)$ the operator investigated by Sălăgean (see [21]).

- $D^n_{\lambda,0}f(z) \equiv D^n_\lambda f(z)$ the operator worked by Al-Oboudi (see [1]).

By the motivation of the paper in [12], we define the following new subclass of analytic functions.

A function $f(z) \in \mathcal{A}$ is said to be in the class $S^n_{\lambda,\mu}(\alpha, t; \delta, A, B)$, if it satisfies for all $z \in U$

$$(1-\alpha)\frac{(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}f(z) - D^n_{\lambda,\mu}f(tz)} + \alpha\frac{\left[(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'\right]'}{\left[D^n_{\lambda,\mu}f(z) - D^n_{\lambda,\mu}f(tz)\right]'} \prec \left(\frac{1+Az}{1+Bz}\right)^\delta,$$

where $|t| \leq 1,\ t \neq 1,\ 0 \leq \alpha \leq 1,\ 0 < \delta \leq 1$ and $-1 \leq B < A \leq 1$.

For special cases of the parameters $n, \lambda, \mu; \alpha, t; \delta, A, B$, we get the following classes introduced and studied by other authors:

- $S^0_{\lambda,\mu}(0, -1; 1, 1, -1)$ by Sakaguchi [20],
- $S^0_{\lambda,\mu}(0, t; 1, 1, -1)$ by Owa et al. [17],
- $S^0_{\lambda,\mu}(1, -1; \delta, A, B)$ by Das and Singh [3],
- $S^0_{\lambda,\mu}(0, -1; 1, A, B)$ by Goel and Mehrok [7],
- $S^0_{\lambda,\mu}(1, -1; 1, A, B)$ by Janteng and Halim [8],
- $S^0_{\lambda,\mu}(\alpha, t; \delta, A, B)$ by Mehrok et al. [12].

Various properties of Sakaguchi type functions and Fekete-Szegö problem for certain classes of univalent functions were discussed by many authors including ([2, 5, 6, 9, 10, 11, 14, 15, 16, 18]). The functions in the class $S^0_{\lambda,\mu}(0, -1; 1, 1, -1) \equiv S^*_s$ are called starlike with respect to symmetrical points and the functions in the class $S^0_{\lambda,\mu}(1, -1; 1, 1, -1) \equiv K_s$ are called convex with respect to symmetrical points in U.

We also introduce the following classes for $|t| \leq 1,\ t \neq 1,\ 0 \leq \alpha \leq 1,$ $0 < \delta \leq 1$ and $-1 \leq B < A \leq 1$:

A function $f(z) \in \mathcal{A}$ is said to be in the class $T^n_{\lambda,\mu}(\alpha, t; \delta, A, B)$ if it satisfies

$$(1-\alpha)\frac{(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}g(z) - D^n_{\lambda,\mu}g(tz)} + \alpha\frac{\left[(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'\right]'}{\left[D^n_{\lambda,\mu}g(z) - D^n_{\lambda,\mu}g(tz)\right]'} \prec \left(\frac{1+Az}{1+Bz}\right)^\delta, g \in S^*_s,$$

and a function $f(z)\in\mathcal{A}$ is said to be in the class $H^n_{\lambda,\mu}(\alpha,t;\delta,A,B)$ if it satisfies

$$(1-\alpha)\frac{(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}h(z)-D^n_{\lambda,\mu}h(tz)}+\alpha\frac{\left[(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'\right]'}{\left[D^n_{\lambda,\mu}h(z)-D^n_{\lambda,\mu}h(tz)\right]'}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta},h\in K_s.$$

The following lemma will be used.

Lemma 1.1. ([20]) *Let* $g(z)=z+\sum\limits_{k=2}^{\infty}b_kz^k\in\mathcal{A}$. *Then, for* $k\geq 2$

$$|b_k|\leq\begin{cases}1, & \text{if } g\in S^*_s\\ \frac{1}{k}, & \text{if } g\in K_s.\end{cases}$$

2. Coefficient Inequalities

For $\lambda\geq\mu\geq 0$, $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $0\leq\alpha\leq 1$, $|t|\leq 1$, $t\neq 1$, $0<\delta\leq 1$ and $-1\leq B<A\leq 1$, we have the following main results.

Theorem 2.1. *If* $f\in S^n_{\lambda,\mu}(\alpha,t;\delta,A,B)$, *then*

$$|a_2|\leq\frac{\delta(A-B)}{(1-t)(1+\alpha)\Phi^n_2},$$

$$|a_3|\leq\begin{cases}\upsilon, & \text{if } |\rho|\leq 1,\\ \upsilon\,|\rho|\,, & \text{if } |\rho|\geq 1,\end{cases}$$

and

$$\left|a_3-\xi a_2^2\right|\leq\begin{cases}\upsilon, & \text{if } |\xi+\sigma|\leq\tau,\\ \frac{\delta^2(A-B)^2}{(1-t)^2(1+\alpha)^2\Phi_2^{2n}}\,|\xi+\sigma|\,, & \text{if } |\xi+\sigma|\geq\tau,\end{cases}$$

where

$$\upsilon=\frac{\delta(A-B)}{(1-t)(2+t)(1+2\alpha)\Phi^n_3},\tag{2.1}$$

$$\rho=B+\frac{(1-\delta)(A-B)}{2}-\frac{\delta(1+t)(1+3\alpha)(A-B)}{(1-t)(1+\alpha)^2},\tag{2.2}$$

$$\tau=\frac{(1-t)(1+\alpha)^2\Phi_2^{2n}}{\delta(2+t)(1+2\alpha)(A-B)\Phi^n_3},\tag{2.3}$$

$$\sigma=\tau\rho.\tag{2.4}$$

P r o o f. Since $f \in S^n_{\lambda,\mu}(\alpha, t; \delta, A, B)$, it follows that

$$(1-\alpha)\frac{(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}f(z)-D^n_{\lambda,\mu}f(tz)}+\alpha\frac{\left[(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'\right]'}{\left[D^n_{\lambda,\mu}f(z)-D^n_{\lambda,\mu}f(tz)\right]'}=\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta}. \tag{2.5}$$

By expanding (2.5), we get

$$\begin{aligned} & 1+(1-t)(1+\alpha)\Phi^n_2 a_2 z \\ & =(1-t)\left[(2+t)(1+2\alpha)\Phi^n_3 a_3-(1+t)(1+3\alpha)\Phi^{2n}_2 a_2^2\right]z^2+... \\ = \quad & 1+\delta(A-B)c_1 z+\delta(A-B)\left(c_2-[B+\frac{1}{2}(1-\delta)(A-B)]c_1^2\right)z^2+.... \end{aligned}$$

From this equality we have

$$a_2 =\frac{\delta(A-B)}{(1-t)(1+\alpha)\Phi^n_2}c_1 \tag{2.6}$$

and

$$a_3=\upsilon\left(c_2-\rho c_1^2\right), \tag{2.7}$$

where υ and ρ are defined in (2.1) and (2.2) respectively. Now, by using (2.6) and (2.7) we obtain

$$a_3-\xi a_2^2 =\upsilon c_2-\frac{\delta^2(A-B)^2}{(1-t)^2(1+\alpha)^2\Phi^{2n}_2}\left[\xi+\sigma\right]c_1^2,$$

where σ is defined in (2.4). By using $|c_1| \leq 1$ from (1.2) we get

$$|a_2| \leq\frac{\delta(A-B)}{(1-t)(1+\alpha)\Phi^n_2}.$$

Applying the well known triangle inequality, we arrive at

$$|a_3| \leq\upsilon\left(|c_2|+|\rho|\,|c_1|^2\right) \tag{2.8}$$

and

$$\left|a_3-\xi a_2^2\right| \leq\upsilon\,|c_2|+\frac{\delta^2(A-B)^2}{(1-t)^2(1+\alpha)^2\Phi^{2n}_2}\,|\xi+\sigma|\,|c_1|^2. \tag{2.9}$$

If we use the inequality $|c_2| \leq 1-|c_1|^2$ in (1.2), the inequalities (2.8) and (2.9) lead us to

$$|a_3| \leq\upsilon\left[1+(|\rho|-1)\,|c_1|^2\right], \tag{2.10}$$

$$\left|a_3 - \xi a_2^2\right| \leq \upsilon + \frac{\delta^2 (A-B)^2}{(1-t)^2 (1+\alpha)^2 \Phi_2^{2n}} \left(|\xi + \sigma| - \tau\right) |c_1|^2 , \tag{2.11}$$

where τ is defined in (2.3).

If $|\rho| \leq 1$, then

$$|a_3| \leq \upsilon$$

and for $|\rho| \geq 1$, by using $|c_1| \leq 1$ from (1.2), the inequality (2.10) implies that

$$|a_3| \leq \upsilon |\rho| .$$

Similarly, for $|\xi + \sigma| \leq \tau$, we have

$$\left|a_3 - \xi a_2^2\right| \leq \upsilon.$$

The bound is sharp for $w(z) = z^2$.

If $|\xi + \sigma| \geq \tau$, again by using $|c_1| \leq 1$ we get

$$\left|a_3 - \xi a_2^2\right| \leq \frac{\delta^2 (A-B)^2}{(1-t)^2 (1+\alpha)^2 \Phi_2^{2n}} |\xi + \sigma| .$$

This bound is sharp for $w(z) = z$. ■

When $n = 0$ and $t = -1$ from Theorem 2.1 we can state the following result given by Mehrok et al. [12].

Corollary 2.2. *If $f \in S^0_{\lambda,\mu}(\alpha, -1; \delta, A, B)$, then*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{\delta(A-B)}{2(1+2\alpha)}, & \text{if } \ |\xi + \sigma_1| \leq \tau_1, \\ \frac{\delta^2 (A-B)^2}{4(1+\alpha)^2} |\xi + \sigma| , & \text{if } \ |\xi + \sigma_1| \geq \tau_1, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &= \frac{(1+\alpha)^2 [2B + (1-\delta)(A-B)]}{(1+2\alpha)\delta(A-B)}, \\ \tau_1 &= \frac{2(1+\alpha)^2}{\delta(1+2\alpha)(A-B)}. \end{aligned}$$

For special cases of the parameters $n, \lambda, \mu; \alpha, t; \delta, A, B,$ in Theorem 2.1 we obtain the following corollaries.

Corollary 2.3. *If $f \in S^*_s$, then*

$$\left|a_3 - \xi a_2^2\right| \leq \max\{1, \ \ |\xi - 1|\} .$$

Corollary 2.4. *If $f \in K_s$, then*

$$\left|a_3 - \xi a_2^2\right| \leq \max\left\{\frac{1}{3},\ \frac{1}{4}\left|\xi - \frac{4}{3}\right|\right\}.$$

Corollary 2.5. *If $f \in S^0_{\lambda,\mu}(0,t;1,1,-1)$, then*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{2}{(1-t)(2+t)}, & \text{if} \quad |\xi + \sigma_2| \leq \tau_2, \\ \frac{4}{(1-t)^2}\left|\xi + \sigma_2\right|, & \text{if} \quad |\xi + \sigma_2| \geq \tau_2, \end{cases}$$

where

$$\begin{aligned} \sigma_2 &= -\frac{3+t}{2(2+t)}, \\ \tau_2 &= \frac{1-t}{2(2+t)}. \end{aligned}$$

Corollary 2.6. *If $f \in S^0_{\lambda,\mu}(1,-1;\delta,A,B)$, then*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{\delta(A-B)}{6}, & \text{if} \quad |\xi + \sigma_3| \leq \tau_3, \\ \frac{\delta^2(A-B)^2}{16}\left|\xi + \sigma_3\right|, & \text{if} \quad |\xi + \sigma_3| \geq \tau_3, \end{cases}$$

where

$$\begin{aligned} \sigma_3 &= \frac{4[2B + (1-\delta)(A-B)]}{3\delta(A-B)}, \\ \tau_3 &= \frac{8}{3\delta(A-B)}. \end{aligned}$$

Corollary 2.7. *If $f \in S^0_{\lambda,\mu}(0,-1;1,A,B)$, then $f \in S^n_{\lambda,\mu}(\alpha,t;\delta,A,B)$*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{\delta(A-B)}{2}, & \text{if} \quad |\xi + \sigma_4| \leq \tau_4, \\ \frac{\delta^2(A-B)^2}{4}\left|\xi + \sigma_4\right|, & \text{if} \quad |\xi + \sigma_4| \geq \tau_4, \end{cases}$$

where

$$\begin{aligned} \sigma_4 &= \frac{2B + (1-\delta)(A-B)}{\delta(A-B)}, \\ \tau_4 &= \frac{2}{\delta(A-B)}. \end{aligned}$$

Corollary 2.8. *If* $f \in S^{0}_{\lambda,\mu}(1,-1;1,A,B)$, *then*

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \frac{A-B}{6}, & \text{if } \ |\xi+\sigma_5| \leq \tau_5, \\ \frac{(A-B)^2}{16}\left|\xi+\sigma_5\right|, & \text{if } \ |\xi+\sigma_5| \geq \tau_5, \end{cases}$$

where

$$\begin{aligned} \sigma_5 &= \frac{8B}{3(A-B)}, \\ \tau_5 &= \frac{8}{3(A-B)}. \end{aligned}$$

Corollary 2.9. *If* $f \in S^{n}_{\lambda,\mu}(0,-1;1,1,-1)$, *then*

$$\left|a_3-\xi a_2^2\right| \leq \max\left\{\frac{1}{\Phi_3^n}, \quad \frac{1}{\Phi_2^{2n}}\left|\xi-\sigma_6\right|\right\}$$

where

$$\sigma_6=\frac{\Phi_2^{2n}}{\Phi_3^n}.$$

Corollary 2.10. *If* $f \in S^{1}_{\lambda,\lambda}(0,-1;1,1,-1)$ *and* $\lambda \geq 0$, *then*

$$\left|a_3-\xi a_2^2\right| \leq \max\left\{\frac{1}{1+6\lambda^2}, \ \frac{1}{(1+2\lambda^2)^2}\left|\xi-\tau_6\right|\right\},$$

where

$$\tau_6=\frac{\left(1+2\lambda^2\right)^2}{1+6\lambda^2}.$$

Corollary 2.11. *If* $f \in S^{1}_{1,1}(0,-1;1,1,-1)$, *then*

$$\left|a_3-\xi a_2^2\right| \leq \max\left\{\frac{1}{7}, \ \frac{1}{9}\left|\xi-\frac{9}{7}\right|\right\}.$$

Corollary 2.12. *If* $f \in S^{1}_{1,0}(0,-1;1,1,-1)$ *then,*

$$\left|a_3-\xi a_2^2\right| \leq \max\left\{\frac{1}{3}, \ \frac{1}{4}\left|\xi-\frac{4}{3}\right|\right\}.$$

Our next result is as follows.

Theorem 2.13. *If $f \in T^n_{\lambda,\mu}(\alpha, t; \delta, A, B)$, then*

$$|a_2| \leq \frac{1}{2}\left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} + (1+t)\right),$$

and

$$|a_3| \leq \begin{cases} \upsilon_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}, & \text{if } \ |\rho_1| \leq 1, \\ \upsilon_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}\,|\rho_1|\,, & \text{if } \ |\rho_1| \geq 1, \end{cases}$$

where

$$\upsilon_1 = \frac{1+t+t^2}{3} + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{3(1+\alpha)(1+2\alpha)\Phi_3^n}, \tag{2.12}$$

$$\rho_1 = B + \frac{1}{2}(1-\delta)(A-B). \tag{2.13}$$

Proof. Since $f \in T^n_{\lambda,\mu}(\alpha, t; \delta, A, B)$, by the definition of subordination

$$(1-\alpha)\frac{(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}g(z) - D^n_{\lambda,\mu}g(tz)} + \alpha\frac{\left[(1-t)z\left(D^n_{\lambda,\mu}f(z)\right)'\right]'}{\left[D^n_{\lambda,\mu}g(z) - D^n_{\lambda,\mu}g(tz)\right]'} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \tag{2.14}$$

for some $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k \in S_s^*$.

By expanding (2.14), we get

$$\begin{aligned} & 1 + (1+\alpha)\Phi_2^n\left[2a_2 - (1+t)b_2\right]z \\ & \Big\{(1+2\alpha)\Phi_3^n\left[3a_3 - \left(1+t+t^2\right)b_3\right] \\ & -(1+t)(1+3\alpha)\Phi_2^{2n}b_2\left[2a_2-(1+t)b_2\right]\Big\}z^2 + ... \\ = \ & 1 + \delta(A-B)c_1 z + \delta(A-B)\left(c_2 - [B + \frac{1}{2}(1-\delta)(A-B)]c_1^2\right)z^2 + ... \end{aligned}$$

After equating the terms, we obtain

$$a_2 = \frac{1}{2}\left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n}c_1 + (1+t)b_2\right), \tag{2.15}$$

$$a_3 = \frac{1+t+t^2}{3}b_3 + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{3(1+\alpha)(1+2\alpha)\Phi_3^n}b_2c_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}\left\{c_2 - \rho_1 c_1\right\}, \tag{2.16}$$

where ρ_1 is defined in (2.13). Applying the triangle inequality, we have

$$|a_2| \leq \frac{1}{2}\left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n}|c_1| + (1+t)|b_2|\right), \tag{2.17}$$

and

$$\begin{aligned} |a_3| \leq \frac{1+t+t^2}{3}|b_3| + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{3(1+\alpha)(1+2\alpha)\Phi_3^n}|b_2||c_1| \\ + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}\{|c_2| + |\rho_1||c_1|\}. \end{aligned} \tag{2.18}$$

By using Lemma 1.1 and the inequalities in (1.2), then (2.17) and (2.18) lead us to

$$|a_2| \leq \frac{1}{2}\left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} + (1+t)\right)$$

and

$$|a_3| \leq \upsilon_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}\left[1 + (|\rho_1| - 1)|c_1|^2\right],$$

where υ_1 is defined in (2.12).

For $|\rho_1| \leq 1$,

$$|a_3| \leq \upsilon_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}$$

and for $|\rho_1| \geq 1$ we have

$$|a_3| \leq \upsilon_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}|\rho_1|.$$

■

Corollary 2.14. *If $f \in T^0_{\lambda,\mu}(\alpha, -1; 1, 1, -1)$, then*

$$|a_2| \leq \frac{1}{1+\alpha},$$

and

$$|a_3| \leq \frac{1}{3} + \frac{2}{3(1+2\alpha)}.$$

Corollary 2.15. *If $f \in T^1_{1,1}(0, -1; 1, 1, -1)$, then*

$$|a_2| \leq \frac{1}{3},$$

and

$$|a_3| \le \frac{3}{7}.$$

Corollary 2.16. *If $f \in T^1_{1,0}(0,-1;1,1,-1)$, then*

$$|a_2| \le \frac{1}{2},$$

and

$$|a_3| \le \frac{5}{9}.$$

Next, we can give the following theorem.

Theorem 2.17. *If $f \in H^n_{\lambda,\mu}(\alpha,t;\delta,A,B)$, then*

$$|a_2| \le \frac{1}{2}\left(\frac{\delta(A-B)}{(1+\alpha)\Phi^n_2} + \frac{1+t}{2}\right),$$

and

$$|a_3| \le \begin{cases} \upsilon_2 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi^n_3}, & \text{if} \quad |\rho_1| \le 1, \\ \upsilon_2 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi^n_3}\left|\rho_1\right|, & \text{if} \quad |\rho_1| \ge 1, \end{cases}$$

where

$$\upsilon_2 = \frac{1+t+t^2}{9} + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi^n_2}{6(1+\alpha)(1+2\alpha)\Phi^n_3},$$

and ρ_1 is defined in (2.13).

References

[1] F. M. Al-Oboudi. On univalent functions defined by a generalized Salagean operator. // *Int. J. Math. Sci.*, **27**, (2004), 1429–1436.

[2] N. E. Cho, O. S. Kwon, S. Owa. Certain subclasses of Sakaguchi functions. // *SEA Bull. Math.*, **17**, (1993), 121–126.

[3] R. N. Das, P. Singh. Radius of convexity for certain subclass of close-to-convex functions. // *J. Indian Math. Soc.*, **41**, (1977), 363–369.

[4] E. Deniz, H. Orhan. The Fekete-Szegö problem for a generalized subclass of analytic functions. // *Kyungpook Math. J.*, **50**, (2010), 37–47.

[5] B. A. Frasin. Coefficient iInequalities for certain classes of Sakaguchi type functions. // *Int. J. Nonlinear Science*, **10**, No 2, (2010), 206–211.

[6] B. A. Frasin, M. Darus, Subordination results on subclasses concerning Sakaguchi functions. // *J. Ineq. and Appl.,* **2009**, (2009), Article ID 574014.

[7] R. M. Goel, B. S. Mehrok. A subclass of starlike functions with respect to symmetric points. // *Tamkang J. of Math.*, **13**, No 1, (1982), 11–24.

[8] A. Janteng, S. A. Halim. A subclass of convex functions with respect to symmetric points. // In: *Proc. 16th National Symp. on Sc. Math.* (2008).

[9] S. Kanas, A. Lecko. On the Fekete-Szegö problem and the domain convexity for a certain class of univalent functions. // *Folia Sci. Univ. Tech. Resov.*, **73**, (1990), 49–57.

[10] S. Kanas, H. E. Darwish. Fekete-Szegö problem for starlike and convex functions of complex order. // *Appl. Math. Letters*, **23**, (2010), 777–782.

[11] S. Kanas, An unified approach to the Fekete-Szegö problem. // *Appl. Math. Comput.*, **218**, (2012), 8453–8461.

[12] B. S. Mehrok, H. Singh, D. Gupta. Coefficient inequality for certain subclasses of analytic functions with respect to symmetric points. // *Gen. Math. Notes*, **3**, No 2, (2011), 88–96.

[13] Z. Nehari. *Conformal Mapping.* McGraw-Hill, New York (1952).

[14] H. Orhan, E. Güneş. Fekete-Szegö inequality for certain subclass of analytic functions. // *General Maths.*, **14**, No 1, (2006), 41–54.

[15] H. Orhan, D. Răducanu. Fekete-Szegö problem for strongly starlike functions associated with generalized hypergeometric functions. // *Math. Comput. Modelling*, **50**, (2009), 430–438.

[16] H. Orhan, N. Yağmur, M. Cağlar. Coefficient estimates for Sakaguchi type functions. // *Sarajevo J. of Math.*, **8**, No 21, (2012), 235–244.

[17] S. Owa, T. Sekine, R. Yamakawa, On Sakaguchi type functions. // *Appl. Math.Comp.*, **187**, (2007), 356–361.

[18] S. Owa, T. Sekine, R. Yamakawa, Notes on Sakaguchi functions. // *RIMS Kokyuroku*, **1414**, (2005), 76–82.

[19] D. Răducanu, H. Orhan, Subclasses of analytic functions defined by a generalized differential operator. // *Int. J. of Math. Anal.*, **4**, No 1, (2010), 1–15.

[20] K. Sakaguchi. On a certain univalent mapping. // *J. Math. Soc. Japan*, **11**, (1959), 72–75.

[21] G. S. Sălăgean, Subclasses of univalent functions. // In: *Compl. Anal. – Proc. 5th Rom.-Finnish Seminar*, Bucharest 1981, Part 1, *Lecture Notes Math.*, **1013**, (1983), 362–372.

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